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SOME RESULTS CONCERNING FRAMES ASSOCIATED WITH MEASURABLE SPACES

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ABSTRACT. In this note some necessary or/and sufficient conditions for the perturbation of a (Ω, μ)-frame are given. We also discussed (Ω, μ)-frames of subspaces.

Keywords: frames, continuous frames, generalized frames.

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1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [8] while addressing some deep problems in non-harmonic Fourier series. Recently, various generalizations of frames have been introduced and studied. Frames of subspaces in Hilbert spaces were first introduced and studied by Casazza and Kutyniok [4] and then by Asgari and Khosravi [3], pseudo frames were introduced by Li and Ogawa [15], oblique frames were first introduced and studied by Eldar [9] and then by Christensen and Eldar [6], outer frames were introduced and studied by Aldourbi, Cabrelli and Molter [1] and Bounded quasi-projectors were studied by Fornasier [11, 12]. Sun [17] introduced a more general concept called G-frames and pointed out that most of the above generalizations of frames may be regarded as a special cases of G-frames and many of their basic properties can be derived within this more general setup.

Another generalization of frames was proposed by Kaiser [14] and independently by Ali Tawreque, Antoine and Gazedu [2] who named it as continuous frames while Kaiser used the terminology generalized frames. Recently, Gabardo and Han [13] studied continuous frames and use the terminology (Ω, μ)-frame.

Discrete and continuous frames arise in many applications in both pure and applied mathematics and, in particular, they play important roles in digital signal processing and scientific computations. For a nice introduction to frames an interested reader may refer to [5] and references therein.

In this note, sufficient condition for the exactness of a (Ω, μ)-frame is obtained. Some necessary and sufficient conditions for the stability of an (Ω, μ)-frame are given. A condition for the perturbation of an (Ω, μ)-frame is obtained. Finally, (Ω, μ)-frames of subspaces are discussed.

2. Preliminaries

Throughout the paper, $\mathcal{H}$ will denote an infinite dimensional Hilbert space. For a family $\{x_\omega\} \subset \mathcal{H}$, $[x_\omega]$ denotes the closure of the $\{x_\omega\}$ in the norm topology of $\mathcal{H}$.

Definition 2.1. *Let (Ω, μ) be a measure space and $\mathcal{H}$ be Hilbert space with inner product. A vector-valued mapping $F : \Omega \to \mathcal{H}$ (i.e. a collection of vectors $F \equiv \{F(\omega)\}_{\omega\in\Omega} \subset \mathcal{H}$) is said to be a* (Ω, μ)-frame *for $\mathcal{H}$ if*

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(1) *F is a weakly measurable function.*
(2) *There exist constants* A *and* B *with* $0 < \mathsf{A} \le \mathsf{B} < \infty$ *such that*

$$\mathsf{A}\|x\|^2 \le \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) \le \mathsf{B}\|x\|^2, \quad x \in \mathcal{H}. \tag{1}$$

The positive constants A and B, respectively, are called lower and upper frame bounds of the (Ω,μ)-frame $F \equiv \{F(\omega)\}_{\omega\in\Omega}$. They are not unique. The inequality (1) is called the (Ω,μ)-*frame inequality.* If $\mathsf{A} = \mathsf{B}$, then $\{F(\omega)\}_{\omega\in\Omega}$ is called *tight* and *normalized tight* if $\mathsf{A} = \mathsf{B} = 1$. The supremum of all A and infimum of all B which satisfy (1) are called *best bounds* for (Ω,μ)-frame. A (Ω,μ)-frame $F \equiv \{F(\omega)\}_{\omega\in\Omega}$ is said to be *exact* if for arbitrary $\Omega_0 \subset \Omega$, with $\mu(\Omega_0) > 0$, $\{F(\omega)\}_{\omega\in\Omega\sim\Omega_0}$ ceases to be a frame for $\mathcal{H}$. If upper inequality of (1) holds then $F \equiv \{F(\omega)\}_{\omega\in\Omega}$ is called a (Ω,μ)-*Bessel family.* The operator $T_F : \mathcal{H} \to L^2(\Omega,\mu)$ defined by

$$(T_F x)(\omega) = \langle x, F(\omega)\rangle, \ \omega \in \Omega, \ x \in \mathcal{H}$$

is bounded linear operator called the *analysis operator* and its conjugate T_F^* is called *synthesis operator* and the operator $T_F^* T_F : \mathcal{H} \to \mathcal{H}$ is called *frame operator* of (Ω,μ)-frame.

A (Ω,μ)-Bessel family $F \equiv \{F(\omega)\}_{\omega\in\Omega}$ is a (Ω,μ)-frame for $\mathcal{H}$ if and only if their exists a (Ω,μ)-Bessel family $G \equiv \{G(\omega)\}$ such that

$$\langle x, y\rangle = \int\limits_{\omega} \langle x, G(\omega)\rangle\langle F(\omega), y\rangle \ d\mu(\omega), \quad \text{for all} \quad x, y \in \mathcal{H}.$$

In this case we say that $\{G(\omega)\}_{\omega\in\Omega}$ is a dual (Ω,μ)-frame for $\{F(\omega)\}_{\omega\in\Omega}$ and $(\{F(\omega)\}, \{G(\omega\})$ a dual pair.

A (Ω,μ)-frame $\{F(\omega)\}_{\omega\in\Omega}$ is *complete* in $\mathcal{H}$ i.e. $\mathcal{H} = [F(\omega)]_{\omega\in\Omega}$.

3. Main results

The following lemma provides a sufficient condition for exactness of (Ω,μ)-frame for a Hilbert space.

Lemma 3.1. *A (Ω,μ)-frame $F \equiv \{F(\omega)\}_{\omega\in\Omega}$ is exact if for arbitrary $\Omega_0 \subset \Omega$ with $\mu(\Omega_0) > 0$, $F(\xi) \notin [F(\omega)]_{\omega\in\Omega\sim\Omega_0}$, for almost all $\xi \in \Omega_0$.*

Proof. Let, if possible, there exist $\Omega_0 \subset \Omega$ with $\mu(\Omega_0) > 0$, $\{F(\omega)\}_{\omega\in\Omega\sim\Omega_0}$ be a (Ω,μ)-frame for $\mathcal{H}$. Then, by frame inequality of $\{F(\omega)\}_{\omega\in\Omega\sim\Omega_0}$, we have $[F(\omega)]_{\omega\in\Omega\sim\Omega_0} = \mathcal{H}$. This gives $F(\xi) \in [F(\omega)]_{\omega\in\Omega\sim\Omega_0}$, for all $\xi \in \Omega$, a contradiction. Hence $F \equiv \{F(\omega)\}_{\omega\in\Omega}$ is exact. □

Now, we show that exact (Ω,μ)-frames are invariant under a linear homeomorphism. An inequality concerning best bounds is also given in the following theorem.

Theorem 3.1. *Let $F \equiv \{F(\omega)\}_{\omega\in\Omega}$ be a (Ω,μ)-frame for $\mathcal{H}$ with best bounds $\mathsf{A}_1, \mathsf{B}_1$ and $U : \mathcal{H} \to \mathcal{H}$ be a linear homeomorphism , then $\{U(F(\omega))\}_{\omega\in\Omega}$ is a (Ω,μ)-frame for $\mathcal{H}$ and its best bounds A_2, B_2 satisfy the inequalities*

$$\mathsf{A}_1\|U\|^{-2} \le A_2 \le A_1\|U^{-1}\|^2,$$
$$\mathsf{B}_1\|U\|^{-2} \le \mathsf{B}_2 \le B_1\|U\|^2.$$

Proof. Since $F : \Omega \to \mathcal{H}$ is weakly measurable i.e. the map $\omega \to \langle F(\omega), x\rangle$ from Ω into $\mathbb{C}$ is measurable for all $x \in \mathcal{H}$. So, the map $\omega \to \langle U(F(\omega)), x\rangle$ from Ω into $\mathbb{C}$ is also measurable for all $x \in \mathcal{H}$.

Now for all $x \in \mathcal{H}$, we have

$$\int\limits_{\Omega} |\langle x, U(F(\omega))\rangle|^2 d\mu(\omega) = \int\limits_{\Omega} |\langle U^*(x), F(\omega)\rangle|^2 d\mu(\omega) \le B_1\|U^*(x)\|^2 \le B_1\|U\|^2\|x\|^2.$$

Also

$$\|x\|^2 = \|UU^{-1}(x)\|^2 \leq \|U\|^2 \ \|U^{-1}(x)\|^2 \leq \frac{\|U\|^2}{\mathsf{A}_1} \int\limits_{\Omega} |\langle U^{-1}(x), F(\omega)\rangle|^2 d\mu(\omega) =$$

$$= \frac{\|U\|^2}{\mathsf{A}_1} \int\limits_{\Omega} |\langle U(U^{-1}(x)), U(F(\omega))\rangle|^2 d\mu(\omega) = \frac{\|U\|^2}{\mathsf{A}_1} \int\limits_{\Omega} |\langle x, U(F(\omega))\rangle|^2 d\mu(\omega).$$

This gives

$$\mathsf{A}_1 \|U\|^{-2} \|x\|^2 \leq \int\limits_{\Omega} |\langle x, U(F(\omega))\rangle|^2 d\mu(\omega), \text{ for all } x \in \mathcal{H}.$$

Therefore

$$\mathsf{A}_1 \|U\|^{-2} \leq A_2, \ \ B_2 \leq B_1 \|U\|^2.$$

Now, for all $x \in \mathcal{H}$, we have

$$\mathsf{A}_2 \|x\|^2 \leq \int\limits_{\Omega} |\langle x, U(F(\omega))\rangle|^2 d\mu(\omega) \leq \mathsf{B}_2 \|x\|^2$$

and

$$\|x\|^2 = \|U^{-1}U(x)\|^2 \leq \|U^{-1}\|^2 \|U(x)\|^2.$$

This gives

$$\mathsf{A}_2 \|U^{-1}\|^{-2} \|x\|^2 \leq \mathsf{A}_2 \|U(x)\|^2 \leq \int\limits_{\Omega} |\langle U(x), U(F(\omega))\rangle|^2 d\mu(\omega) \ \left(= \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) \right) \leq$$

$$\leq \mathsf{B}_2 \|U(x)\|^2 \leq \mathsf{B}_2 \|U\|^2 \ \|x\|^2, \text{for all } x \in \mathcal{H}.$$

Therefore, we have

$$\mathsf{A}_2 \|U^{-1}\|^{-2} \leq A_1, \ \ \ B_1 \leq \mathsf{B}_2 \ \|U\|^2.$$

Hence

$$\begin{aligned} \mathsf{A}_1 \|U\|^{-2} &\leq A_2 \leq \mathsf{A}_1 \|U^{-1}\|^2, \\ \mathsf{B}_1 \|U\|^{-2} &\leq \mathsf{B}_2 \leq \mathsf{B}_1 \|U\|^2. \end{aligned}$$

Corollary 3.1. *If $\{F(\omega)\}_{\omega \in \Omega}$ is exact, then so is $\{U(F(\omega))\}_{\omega \in \Omega}$.*

The following theorem gives a necessary and sufficient condition for the perturbation of a (Ω, μ)-frame.

Theorem 3.2. *Let $\{F(\omega)\}_{\omega \in \Omega}$ be a (Ω, μ)-frame for a Hilbert space $\mathcal{H}$ and $G : \Omega \to \mathcal{H}$ be a vector-valued function. Then the following statements are equivalent:*

(1) *$\{G(\omega)\}_{\omega \in \Omega}$ is a (Ω, μ)-frame for $\mathcal{H}$.*

(2) *there exists $M > 0$ such that*

$$\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) \leq M \min \left\{ \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega), \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega) \right\}.$$

(3) *There exists $K > 0$ such that*

$$\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) \leq K \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega).$$

Proof. *(i)* $\Rightarrow$ *(ii)* Let $\mathsf{A}_F, \mathsf{B}_F$ and $\mathsf{A}_G, \mathsf{B}_G$ be frame bounds for the (Ω, μ)-frames $\{F(\omega)\}_{\omega\in\Omega}$ and $\{G(\omega)\}_{\omega\in\Omega}$ respectively. Then, for all $x \in \mathcal{H}$, we have

$$\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) = \int\limits_{\Omega} |\langle x, F(\omega)\rangle - \langle x, G(\omega)\rangle|^2 d\mu(\omega) \leq$$

$$\leq 2\bigg(\int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq 2\bigg(\int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) + \mathsf{B}_G \, \|x\|^2\bigg) \leq$$

$$\leq 2\bigg(\int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) + \frac{\mathsf{B}_G}{\mathsf{A}_F} \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega)\bigg) = 2\bigg(1 + \frac{\mathsf{B}_G}{\mathsf{A}_F}\bigg) \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega).$$

Similarly, we can show that

$$\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) \leq 2\bigg(1 + \frac{\mathsf{B}_F}{\mathsf{A}_G}\bigg) \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega).$$

(ii) $\Rightarrow$(i) For all $x \in \mathcal{H}$, we have

$$\mathsf{A}_F\|x\|^2 \leq \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 \leq 2\bigg(\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq$$

$$\leq 2\bigg(M \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq 2(M+1)\bigg(\int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq$$

$$\leq 4(M+1)\bigg(\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq 4(M+1) \times$$

$$\times\bigg(M \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega)\bigg) = 4(M+1)^2 \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 d\mu(\omega) \leq$$

$$\leq 4(M+1)^2 \mathsf{B}_F \|x\|^2.$$

This gives

$$\frac{\mathsf{A}_F}{2(1+M)} \, \|x\|^2 \leq \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega) \leq \; 2(M+1) \; \mathsf{B}_F \; \|x\|^2 \,, \; x \in \mathcal{H}.$$

Hence $\{G(\omega)\}_{\omega\in\Omega}$ is a (Ω, μ)-frame for $\mathcal{H}$.

(ii)$\Rightarrow$ (iii) is clear.

(iii)$\Rightarrow$ (i) Since

$$\mathsf{A}_F \; \|x\|^2 \leq \int\limits_{\Omega} |\langle x, F(\omega)\rangle|^2 \leq 2\bigg(\int\limits_{\Omega} |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq$$

$$\leq 2\bigg(K \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega) + \int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) = 2(K+1)\bigg(\int\limits_{\Omega} |\langle x, G(\omega)\rangle|^2 d\mu(\omega)\bigg) \leq$$

$$\leq 2(K+1)\|x\|^2, \quad \text{for all } x \in \mathcal{H}.$$

Hence $\{G(\omega)\}_{\omega\in\Omega}$ is a (Ω, μ)-frame for $\mathcal{H}$. ☐

Now, we give a sufficient condition for perturbation of an (Ω, μ)-frame.

Theorem 3.3. *Let $\{F(\omega)\}_{\omega\in\Omega}$ be a (Ω,μ)-frame for $\mathcal{H}$ and $z_0\in\mathcal{H}$ such that $\langle z_0, F(\omega)\rangle = \lambda$, for all $\omega\in\Omega$, where λ is non-zero scalar. Then,*

(1) *there exists a non-zero vector $v\in\mathcal{H}$ such that $\{F(\omega)+v\}_{\omega\in\Omega}$ is not a (Ω,μ)-frame for $\mathcal{H}$.*
(2) *for each $\xi\in\Omega$, there exists a non-zero vector $Z_\xi\in\mathcal{H}$ such that $\{F(\omega)+Z_\xi\}_{\omega\in\Omega}$ is not a (Ω,μ)-frame for $\mathcal{H}$.*

Proof. (1) Choose a vector $x\in\mathcal{H}$ (which may be equal to z_0) such that $\langle z_0, x\rangle = \alpha$, where α is a non-zero scalar. Put $v = -\overline{(\frac{\lambda}{\alpha})}x$. Then, v is a non-zero vector in $\mathcal{H}$ such that $\{F(\omega)+v\}_{\omega\in\Omega}$ is not a (Ω,μ)-frame for $\mathcal{H}$. Indeed, let $0<\mathsf{A}\le\mathsf{B}<\infty$ be positive constants such that

$$\mathsf{A}\|x\|^2 \le \int_\Omega |\langle x, F(\omega)+v\rangle|^2 d\mu(\omega) \le \mathsf{B}\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

Then, in particular for $x=z_0$, we have

$$\mathsf{A}\|z_0\|^2 \le \int_\Omega |\langle z_0, F(\omega)+v\rangle|^2 d\mu(\omega) \le \mathsf{B}\|z_0\|^2.$$

Now, for all $\omega\in\Omega$, we have
$\langle z_0, F(\omega)+v\rangle = \langle z_0, F(\omega)\rangle + \langle z_0, v\rangle = \lambda + \langle z_0, -\overline{(\frac{\lambda}{\alpha})}x\rangle = 0$. By lower inequality, we obtain $z_0=0$. This is a contradiction.

(2) Fix $\xi\in\Omega$. Put $Z_\xi = -F(\xi)$. Then, Z_ξ is a non-zero vector in $\mathcal{H}$ such that $\{F(\omega)+Z_\xi\}_{\omega\in\Omega}$ is not a (Ω,μ)-frame for $\mathcal{H}$. $\square$

Let $\{F(\omega)\}_{\omega\in\Omega}$ be a (Ω,μ)-frame for $\mathcal{H}$ and $G:\Omega\to\mathcal{H}$ be a vector-valued function such that $\{F(\omega)-G(\omega)\}_{\omega\in\Omega}$ be a (Ω,μ)-Bessel family. Then, in general, $\{G(\omega)\}_{\omega\in\Omega}$ is not a (Ω,μ)-frame for $\mathcal{H}$. The reason is that Bessel bound for $\{F(\omega)-G(\omega)\}_{\omega\in\Omega}$ is not less that lower bound for the frame $\{F(\omega)\}_{\omega\in\Omega}$ or the following inequality

$$\begin{aligned}\int_\Omega |\langle x, F(\omega)\rangle|^2 d\mu(\omega) - \int_\Omega |\langle x, G(\omega)\rangle|^2 d\mu(\omega) \ &\le\ \gamma\Big(\int_\Omega |\langle x, F(\omega)\rangle|^2 d\mu(\omega) - \\ - \int_\Omega |\langle x, G(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega |\langle x, F(\omega)-G(\omega)\rangle|^2 d\mu(\omega)\Big), &\quad \text{for some } \gamma\ge 2. \qquad (2)\end{aligned}$$

is not satisfied. In this direction we have

Theorem 3.4. *Let $\{F(\omega)\}_{\omega\in\Omega}$ be a (Ω,μ)-frame for $\mathcal{H}$ with the bounds A,B and a vector-valued function $G:\Omega\to\mathcal{H}$ such that $\{F(\omega)-G(\omega)\}_{\omega\in\Omega}$ is a (Ω,μ)-Bessel family for $\mathcal{H}$ with bound $\mathsf{M}<\mathsf{A}$, such that (2) holds. Then $\{G(\omega)\}_{\omega\in\Omega}$ is a (Ω,μ)-frame for $\mathcal{H}$. Conversely, if $\{F(\omega)\}_{\omega\in\Omega}$ and $\{G(\omega)\}_{\omega\in\Omega}$ are (Ω,μ)-frames for $\mathcal{H}$ with bounds $\mathsf{A}_1,\mathsf{B}_1$ and $\mathsf{A}_2,\mathsf{B}_2$ respectively, and $U:\mathcal{H}\to\mathcal{H}$ is a linear homeomorphism such that $U(F(\omega)) = G(\omega)$, $\omega\in\Omega$, then $\{F(\omega)-G(\omega)\}_{\omega\in\Omega}$ is a (Ω,μ)-Bessel family for $\mathcal{H}$ with best bound $M=\min\{\mathsf{B}_1\|I-U\|^2, \mathsf{B}_2\|I-U^{-1}\|^2\}$.*

Proof. A simple calculation show that $\{G(\omega)\}_{\omega\in\Omega}$ is an (Ω,μ)-frame for $\mathcal{H}$.

Conversely, since

$$\int_\Omega |\langle x, F(\omega) - G(\omega)|^2 d\mu(\omega) = \int_\Omega |\langle x, F(\omega)\rangle - \langle x, U(F(\omega))\rangle|^2 d\mu(\omega) =$$

$$= \int_\Omega |\langle (I - U^*)x, F(\omega)\rangle|^2 d\mu(\omega) \leq \mathsf{B}_1 \|(I - U^*)x\|^2 \leq \mathsf{B}_1 \|I - U\|^2 \|x\|^2.$$

and

$$\int_\Omega |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\langle x, U^{-1}(G(\omega)) - G(\omega)|^2 d\mu(\omega) =$$

$$= \int_\Omega |\langle (U^{-1} - I)^* x, G(\omega)\rangle|^2 d\mu(\omega) \leq \mathsf{B}_2 \|(U^{-1} - I)^* x\|^2 \leq \mathsf{B}_2 \|I - U^{-1}\|^2 \|x\|^2, \text{ for all } x \in \mathcal{H}.$$

Hence

$$\int_\Omega |\langle x, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) \quad \leq \quad M = \min\{\mathsf{B}_1 \|I - U\|^2, \mathsf{B}_2 \|I - U^{-1}\|^2\} \|x\|^2.$$

□

Remark 3.1. *Let $\{F(\omega)\}_{\omega\in\Omega}$ be an (Ω, μ)-frame for $\mathcal{H}$ and $\{G(\omega)\}_{\omega\in\Omega}$ be an (Ω, μ)-Bessel family in $\mathcal{H}$ (with bound M). Then, in general, $\{F(\omega) + \lambda G(\omega)\}_{\omega\in\Omega}$ is not an (Ω, μ)-frame for $\mathcal{H}$, where λ is some scalar. However under certain conditions, namely $|\lambda| < \sqrt{\frac{\mathsf{A}}{\mathsf{M}}}$ and*

$$\int_\Omega |\langle x, F(\omega)\rangle|^2 d\mu(\omega) - \int_\Omega |\langle x, \lambda G(\omega)\rangle|^2 d\mu(\omega) \leq \gamma \Bigg(\int_\Omega |\langle x, F(\omega)\rangle|^2 d\mu(\omega) +$$

$$+ \int_\Omega |\langle x, G(\omega)\rangle|^2 d\mu(\omega) - \int_\Omega |\langle x, F(\omega) - \lambda G(\omega)\rangle|^2 d\mu(\omega) \Bigg), \text{ for some } \gamma \geq 2,$$

the collection $\{F(\omega) - \lambda G(\omega)\}_{\omega\in\Omega}$ turns out to be a (Ω, μ)-frame for $\mathcal{H}$.

4. (Ω, μ)-FRAMES OF SUBSPACES

Definition 4.1. *Let Ω be a measure space with positive measure μ and $\{v_\omega\}_{\omega\in\Omega}$ be a family of weights, i.e., $v_\omega > 0$ for all $\omega \in \Omega$. For each $\omega \in \Omega$, $\pi_{W_\omega} : \mathcal{H} \to W_\omega$ denote the projection of $\mathcal{H}$ onto W_ω. A family of closed subspaces $\{W_\omega\}_{\omega\in\Omega}$ of a Hilbert space $\mathcal{H}$ is a* (Ω, μ)-frame of subspaces *with respect to $\{v_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ if*

(1) *for each $x \in \mathcal{H}$, $\omega \to \|\pi_{W_\omega}(x)\|$ is a measurable function on Ω.*
(2) *there exist constants A and B with $0 < \mathsf{A} \leq \mathsf{B} < \infty$ such that*

$$\mathsf{A}\|x\|^2 \leq \int_\Omega v_\omega^2 \|\pi_{W_\omega}(x)\|^2 d\mu(\omega) \leq \mathsf{B}\|x\|^2, \quad x \in \mathcal{H}. \tag{3}$$

The constants A and B are called (Ω, μ)-*frame bounds* for the (Ω, μ)-frame of subspaces. The (Ω, μ)-frame of subspaces $\{W_\omega\}_{\omega\in\Omega}$ with respect to $\{v_\omega\}_{\omega\in\Omega}$ is said to be *tight*, if in inequality (3) the constants A and B can be chosen so that $\mathsf{A} = \mathsf{B}$. It is called *Parseval* (Ω, μ)-frame of subspaces with respect to $\{v_\omega\}_{\omega\in\Omega}$ provided $\mathsf{A} = \mathsf{B} = 1$. The family $\{W_\omega\}_{\omega\in\Omega}$ is called a (Ω, μ)-*Bessel family of subspaces* with respect to $\{v_\omega\}_{\omega\in\Omega}$ with (Ω, μ)-Bessel bound B if it satisfies the upper inequality in (3).

Definition 4.2. *A family $\{x_\omega\}_{\omega\in\Omega} \subset \mathcal{H}$ is said to be a (Ω, μ)-frame family for $\mathcal{H}$ if $\{x_\omega\}_{\omega\in\Omega}$ is a (Ω, μ)-frame for $[x_\omega]_{\omega\in\Omega}$.*

The following theorem gives necessary and sufficient condition for a family of closed subspaces of a Hilbert space to be a (Ω, μ)-frame of subspaces

Theorem 4.1. *For each $\omega \in \Omega$, let $v_\omega > 0$ and let $\{N_\omega\}_{\omega\in\Omega}$ be a family of disjoint subspaces of Ω such that $\bigcup\limits_{\omega\in\Omega} N_\omega = \Omega$. For each $\omega \in \Omega$, let $\{x_{j\omega}\}_{j\in N_\omega}$ be a (Ω, μ)-frame family with (Ω, μ)-frame family bounds A_ω and B_ω. Define $W_\omega = \overline{\operatorname{span}}_{j\in N_\omega}\{x_{j\omega}\}$ for all $\omega \in \Omega$. Suppose that $0 < \mathsf{A} = \inf_{\omega\in\Omega} \mathsf{A}_\omega \le \mathsf{B} = \sup_{\omega\in\Omega} \mathsf{B}_\omega < \infty$. Then $\{v_\omega x_{j\omega}\}_{j\in N_\omega, \omega\in\Omega}$ is a (Ω, μ)-frame for $\mathcal{H}$ if and only if $\{W_\omega\}_{\omega\in\Omega}$ is a (Ω, μ)-frame of subspaces of $\mathcal{H}$.*

Proof. Since for each $\omega \in \Omega$, $\{x_{j\omega}\}_{j\in W_\omega}$ is a (Ω, μ)-frame for N_ω with (Ω, μ)-frame bounds A_ω and B_ω. So, for each $x \in \mathcal{H}$

$$\mathsf{A} \int\limits_{\omega\in\Omega} v_\omega^2 \|\pi_{W_\omega}(x)\|^2 d\mu(\omega) \le \int\limits_{\omega\in\Omega} \mathsf{A}_\omega v_\omega^2 \|\pi_{W_\omega}(x)\|^2 d\mu(\omega) \le$$

$$\le \int\limits_{\omega\in\Omega} \int\limits_{j\in N_\omega} |\langle \pi_{W_\omega}(x), v_\omega x_{j\omega}\rangle|^2 d\mu(j) d\mu(\omega) \le B \int\limits_{\omega\in\Omega} v_\omega^2 \|\pi_{W_\omega}(x)\|^2 d\mu(\omega),$$

by hypothesis

$$\int\limits_{\omega\in\Omega} \int\limits_{j\in N_\omega} |\langle \pi_{W_j}(x), v_\omega x_{j\omega}\rangle|^2 d\mu(j) d\mu(\omega) = \int\limits_{\omega\in\Omega} \int\limits_{j\in N_\omega} |\langle x, v_\omega x_{j\omega}\rangle|^2 d\mu(j) d\mu(\omega).$$

Hence, we conclude that if $\{v_\omega x_{j\omega}\}_{j\in N_\omega, \omega\in\Omega}$ is a (Ω, μ)-frame for $\mathcal{H}$ with bounds C and D, then the collection $\{W_\omega\}_{\omega\in\Omega}$ form a (Ω, μ)-frame of subspaces with respect to $\{v_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with frame bound C/B and D/A. $\square$

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